A new stochastic renormalization approach to random processes with very long memory: fractal time as a process with 'almost' complete connections

1992 J. Phys. A: Math. Gen. 25 749

(http://iopscience.iop.org/0305-4470/25/4/014)

J. Phys. A: Math. Gen. 25 (1992) 749–753. Printed in the UK

A new stochastic renormalization approach to random processes with very long memory: fractal time as a process with 'almost' complete connections

Marcel Ovidiu Vlad†

Institut für Festkörperforschung, Forschungszentrum Jülich, W 5170 Jülich, Federal Republic of Germany

Received 18 July 1991

Abstract. An attempt is made to apply the mathematical theory of random jump processes with complete connections to the physically oriented description of very long memory effects. A new stochastic renormalization approach is introduced. Starting from a stochastic process with infinite memory, we attach to each step a very small probability that the memory is lost. As a result of this change the memory acts over a variable number of steps, forming 'blocks' of different sizes. This renormalization procedure leads to the self-similarity of the probability of the number of steps over which the memory is kept. The model yields a new explanation for the fractal time.

The random processes with complete connections (RPCC) were introduced by mathematicians a long time ago (Onicescu and Mihoc 1937). Such processes have been applied to many problems in biology and psychology (Iosifescu and Grigorescu 1989); they are almost unknown within the theoretical physics community. Physicists, however, are very interested in the description of memory effects merely in connection with polymer physics, normal and exotic diffusion, dynamics of growth processes, diffusion-limited aggregation, kinetic critical phenomena, etc. (Haus and Kehr 1978, 1979, Kutner 1985, Pietronero and Sibesma 1987, Peliti and Pietronero 1987, Shlesinger and Klafter 1989). At first sight it would seem that the RPCC theory is of little interest to physicists. It is a rather formal theory based on the assumption that a system remembers all its previous history. By assuming that a system is described by a set of random variables $\boldsymbol{X} = (X^{(1)}, X^{(2)}, \ldots)$ the dynamics of the process is described by a succession of jump events $\boldsymbol{X}_0 \to \boldsymbol{X}_1 \to \boldsymbol{X}_2 \to \boldsymbol{X}_3 \to \boldsymbol{X}_4 \to \ldots$. The probability $T_n \, \mathrm{d}\boldsymbol{X}_n$ of transition between two successive states $\boldsymbol{X}_{n-1}, \boldsymbol{X}_n$ depends on $\boldsymbol{X}_{n-1}$, $\boldsymbol{X}_n$ as well as on all preceding states $\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_{n-1}$:

$$T_n \, \mathrm{d}\boldsymbol{X}_n = T_n(\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_{n-1} \to \boldsymbol{X}_n) \, \mathrm{d}\boldsymbol{X}_n \tag{1}$$

$$\int T_n \, \mathrm{d}\boldsymbol{X}_n = 1. \tag{2}$$

From the physical point of view, however, it is more plausible to assume that not all physical realizations of the process have infinite memory. This fact is suggested by

† Alexander von Humboldt fellow. Permanent address: C M Bucuresti, Casuta Postala 77-49, Oficiul Bucuresti 77, Romania.

the scaling behaviour displayed by many physical phenomena (Pietronero and Sibesma 1987, Peliti and Pietronero 1987). That is the reason why we shall introduce a new type of stochastic process by assuming that for each jump there is a very small probability μ that the memory is lost. The complementary value $\lambda=1-\mu$ is the probability that the memory is kept. This change defines a kind of 'stochastic renormalization' which leads to a self-similar behaviour. In this case the different realizations of the process consist of 'blocks' of jumps over which the memory is kept. The number of jumps from a given block is itself a random variable. Thus, we can introduce the joint probability

$$\Phi_n(\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n; q)\, \mathrm{d}\boldsymbol{X}_0\, \mathrm{d}\boldsymbol{X}_1 \ldots \mathrm{d}\boldsymbol{X}_n \tag{3}$$

that after q jump events the memory acts over n preceding states and that the corresponding state vectors are between $\boldsymbol{X}_0$ and $\boldsymbol{X}_0+\mathrm{d}\boldsymbol{X}_0, \ldots, \boldsymbol{X}_n$ and $\boldsymbol{X}_n+\mathrm{d}\boldsymbol{X}_n$. By taking the above hypotheses into account we can deduce the evolution equation:

$$\begin{aligned}\Phi_n(\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n; q) &\\ = (1-\delta_{n0})\lambda_n T_n(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_{n-1} \to \boldsymbol{X}_n)\Phi_{n-1}(\boldsymbol{X}_0, \ldots, \boldsymbol{X}_{n-1}; q-1) &\\ +\delta_{n0} \sum_{n'\geqslant 0} (1-\lambda_{n'+1}) \int \ldots \int \Phi_{n'}(\boldsymbol{X}'_0, \ldots, \boldsymbol{X}'_{n'}; q-1) &\\ \times T_{n'+1}(\boldsymbol{X}'_0, \ldots, \boldsymbol{X}'_{n'} \to \boldsymbol{X}_0)\, \mathrm{d}\boldsymbol{X}'_0 \ldots \mathrm{d}\boldsymbol{X}'_{n'} &\end{aligned} \tag{4}$$

where $\lambda=\lambda_n$ is the probability that at the nth jump from a block the memory is kept. Equation (4) could be considered as the renormalized expression corresponding to an RPCC. We note that for $\lambda_n=1$ and $q=n$ we recover the non-renormalized case. Solving equation (4) is rather difficult and beyond the scope of the present paper. The detailed analysis of this equation will be the subject of our future investigations. Some general features of the model can be investigated without solving the equation (4). We introduce the probability

$$\xi_n(q) = \int \ldots \int \Phi_n\, \mathrm{d}\boldsymbol{X}_0 \ldots \mathrm{d}\boldsymbol{X}_n \tag{5}$$

that after q jumps the memory acts over n preceding steps. By integrating each term of equation (4) over $\boldsymbol{X}_0, \ldots, \boldsymbol{X}_n$ we get a closed equation in $\xi_n(q)$:

$$\xi_n(q) = (1-\delta_{n0})\lambda_n \xi_{n-1}(q-1) + \delta_{n0} \sum_{n'\geqslant 0} (1-\lambda_{n'+1})\xi_{n'}(q-1). \tag{6}$$

Unlike the initial equation in Φ_n the reduced equation (6) is Markovian. It is easy to show that the solution $\xi_n(q)$ of equation (6) evolves towards a persistent form:

$$\xi_n(q) \to \xi_n^{\mathrm{st}} \qquad \text{as } q \to \infty \tag{7}$$

where

$$\xi_n^{\mathrm{st}} = (1-\lambda_{n+1})\lambda_n \ldots \lambda_1 \qquad \xi_0^{\mathrm{st}} = 1-\lambda_1. \tag{8}$$

The physical interpretation of equation (8) is clear. The persistent probability ξ_n^{st} can be expressed as a product of the probabilities $\lambda_1, \ldots, \lambda_n$ and $1-\lambda_{n+1}$ that the memory is kept after the first n jumps and is lost after the occurrence of the $(n+1)$ one.

The renormalization procedure depends on the choice of probabilities $\lambda_1, \lambda_2, \ldots$. We asume that the block generation takes place in a hierarchical manner, i.e. that the elementary jumps are lumped into blocks, the blocks into blocks of blocks, etc. The number l of successive lumping events generating a certain block is itself random. Denoting by c the probability that the memory is kept after a lumping event λ_n can be evaluated as a mean over all possible values of the lumping events:

$$\lambda_n = \sum_{l=1}^{\infty} c^l \varphi_l(n) \tag{9}$$

where $\varphi_l(n)$ is the probability that a block of n jumps is the result of l lumping events. Denoting by b the probability that a lumping event corresponds to an elementary jump from a given block, we have

$$\varphi_l(n) = (1-b^n)(b^n)^{l-1}. \tag{10}$$

By evaluating the sum (9) we get

$$\lambda_n(c, b) = c(1-b^n)/(1-cb^n). \tag{11}$$

As we are interested in the study of very long memory we shall consider the limit $c, b \to 1$. We shall assume that the ratio

$$H = \ln c/\ln b \tag{12}$$

remains constant. We note that H is a kind of 'fractal exponent' attached to the lumping process. We get

$$\lambda_n = \lim_{c,b\to 1, H=\mathrm{const}} \lambda_n(c, b) = n/(n+H). \tag{13}$$

Inserting equation (13) into equation (8) we get the following expression for ξ_n^{st}:

$$\xi_n^{\mathrm{st}} = H\Gamma(H+1)n!/\Gamma(H+n+2) \tag{14}$$

where $\Gamma(z) = \int_0^\infty y^{z-1} \exp(-y)\, \mathrm{d}y$ is Euler's complete gamma function. If $1 > H > 0$ all moments $\langle n \rangle, \langle n^2 \rangle, \ldots$, corresponding to equation (14) are infinite and the asymptotic behaviour of ξ_n^{st} is described by a statistical fractal

$$\xi_n^{\mathrm{st}} \simeq H\Gamma(H+1)n^{-(1+H)} \qquad \text{as } n \to \infty. \tag{15}$$

The above considerations can be extended to the more general case when the lumping process starts from small blocks of random size n_0 described by a certain probability law $\eta(n_0)$, where all moments of n_0 are finite. The above-mentioned situation corresponds to $\eta(n_0) = \delta_{n_0 1}$, i.e. the lumping process starts from isolated jumps. For an arbitrary distribution $\eta(n_0)$ the resulting distribution ξ_n^{st} can be expressed as a sum over all possible values of n_0:

$$\begin{aligned} \xi_n^{\mathrm{st}} &= \sum_{n_0 \geq 1} \eta(n_0)\lambda_{n_0} \ldots \lambda_n(1-\lambda_{n+1}) \\ &= \sum_{n_0 \geq 1} \eta(n_0) H\Gamma(H+n_0)n!/(\Gamma(H+n+2)(n_0-1)!). \end{aligned} \tag{16}$$

The asymptotic behaviour is similar to that predicted by equation (14). For $1 > H > 0$ all moments $\langle n \rangle, \langle n^2 \rangle, \ldots$ are infinite and we have

$$\xi_n^{\mathrm{st}} \simeq Hn^{-(1+H)} \sum_{n_0 \geq 1} \eta(n_0)\Gamma(H+n_0)/(n_0-1)! \qquad \text{as } n \to \infty. \tag{17}$$

Now we can evaluate the distribution $\psi^{\mathrm{st}}(t)$ of the time interval t within which the memory is kept. For simplicity we shall assume that all jumps take place with the frequency Ω, irrespective of the values of the state vectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$. It turns out that the time between two jumps is exponentially distributed and $\psi^{st}(t)$ is simply equal to

$$\psi^{st}(t) = \sum_{n=0}^{\infty} \xi_n^{st}(\Omega \exp(-\Omega t)\otimes)^{n+1} \tag{18}$$

where $\otimes$ denotes the temporal convolution product. After some calculus we get

$$\psi^{st}(t) = H\Omega^{-H} t^{-(1+H)} \sum_{n_0 \geq 1} [(n_0-1)!]^{-1} \eta(n_0) \gamma(H+n_0, \Omega t) \tag{19}$$

where $\gamma(z, x) = \int_0^x y^{z-1} \exp(-y)\, dy$ is the incomplete gamma function. As expected, for $1 > H > 0$ the asymptotic behaviour is described by a statistical fractal,

$$\gamma^{st}(t) \simeq H\Omega^{-H} t^{-(H+1)} \sum_{n_0 \geq 1} [(n_0-1)!]^{-1} \eta(n_0) \Gamma(H+n_0) \qquad \text{as } t \to \infty \tag{20}$$

whereas for $t \to 0$ we get an exponential dependence modulated by a polynomial in t

$$\psi^{st}(t) \simeq e^{-\Omega t} H \sum_{n_0 \geq 1} [(n_0-1)!]^{-1} \eta(n_0) (H+n_0)^{-1} \Omega^{n_0} t^{n_0-1} \qquad \text{as } t \to 0. \tag{21}$$

The above self-similar construction of the probabilities λ_n leads to a statistical fractal behaviour for the number of steps n or the time t over which the memory acts. The physical significance of this self-similarity is clear: there are many blocks of jumps ranging from very small to very large ones. For low n, λ_n may be relatively small whereas, as $n \to \infty$, λ_n increases towards unity. The compensation of these two opposite factors generates a large variety of blocks leading to the lack of a characteristic memory scale.

Within this paper we have analysed only the way in which the memory is lost. However, the self-similar structure of the model leads to a self-similar behaviour of the joint probability densities. Although small, the loss of the memory allows the joint probability densities to evolve towards time-persistent forms, even if this is not true for the non-renormalized model. We can distinguish a discrete time description, which can be done in terms of equation (4), or a continuous time description, which could be analysed by generalizing the multistep analogue of the continuous-time random-walk theory (Shlesinger and Klafter 1989) or the method of age-dependent master equations (Vlad and Pop 1989). Work on this problem is in progress and will be presented elsewhere.

Another problem which should be clarified is the relationship with other stochastic renormalization theories. Recently, we have introduced a new class of stochastic renormalization approaches (Vlad 1992) based on an assumption similar to the one used here: the renormalization is described as a clustering of jumps. Although the two ideas seem to be similar, there are many differences. The above-mentioned formalism is not directly related to the long memory and the distribution of the cluster size is evaluated in a different way. The predicted results are also different. In particular, unlike the case of our approach the asymptotic behaviour of cluster size distribution is described by an inverse power law modulated by a periodic function of the logarithm of the cluster size. The logarithmic oscillations occur frequently within the framework of stochastic renormalization (Cassandro and Jona Lasinio 1978, Shlesinger and Hughes 1981). At the present stage of our research it is not clear why they are absent in the case of our model.

Acknowledgments

The author wishes to thank Professor K W Kehr for helpful suggestions. Thanks are also due to the Alexander von Humboldt Foundation for financial support and to the Forschungszentrum Jülich for hospitality.

References

Cassandro M and Jona Lasinio G 1978 *Adv. Phys.* **27** 913
Haus J W and Kehr K W 1978 *Solid State Commun.* **26** 753
—— 1979 *J. Phys. Chem. Solids.* **40** 1019
Kutner R 1985 *J. Phys. C.: Solid State Phys.* **18** 6323
Iosifescu M and Grigorescu S 1989 *Dependence with Complete Connections and Applications* (Cambridge: Cambridge Tracts)
Onicescu O and Mihoc G 1937 *La Dépendence Statistique, Chaînes et Familles de Chaînes Discontinues* (Paris: Hermann) (in *Sér. Actualités Scientifiques et Industrielles* **503**)
Peliti L and Pietronero K 1987 *Riv. Nuovo Cimento* **10**(6) 1
Pietronero L and Sibesma A P 1987 *Europhys. Lett.* **4** 869
Shlesinger M F and Klafter J 1989 *J. Phys. Chem.* **93** 7023
Vlad M O 1992 *Phys. Rev.* A in press
Vlad M O and Pop A 1989 *Physica* **159A** 267